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Density of strings formed at a second-order cosmological phase transition

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Abstract

We discuss the problem of estimating the characteristic length scale ξ_s , and hence the initial density, of a system of cosmic strings formed at a continuous, second-order phase transition in the early universe. In particular, we examine the roles of the correlation length (or string width) ξ_c and the Ginzburg length ξ_G which defines the “fuzziness” of long strings. We argue that strings acquire a clear identity only once ξ_s exceeds both ξ_c and ξ_G , and estimate its magnitude at that time.

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I. INTRODUCTION

Strings or vortices may be formed at phase transitions in condensed-matter systems and in the early universe [1]. An important problem in either case is to estimate the initial vortex density just after the transition. It has recently been recognized that previously accepted ideas about this are wrong in important ways, so a re-examination of the question is timely. In the present note, we shall confine our attention to the case of a continuous, second-order transition, involving the breaking of a global Abelian $U(1)$ symmetry. (We believe the results would be similar in the case of a local, gauge symmetry.)

For many purposes, at least in cosmology, the precise nature of the initial conditions is not important. The strings apparently evolve towards a scaling solution that is largely independent of the initial conditions. Nevertheless, for some applications, for example in discussions of string-mediated baryogenesis [2], and certainly in the condensed-matter field, it is important. In any case, it is important to know that the initial defect density is not essentially zero.

The evolution of cosmic strings has been extensively studied by numerical simulation. The initial conditions for these studies have been set by choosing random phases on some lattice of points and using the “geodesic rule” to decide whether a string passes through each plaquette [3]. Along each edge of the lattice the phase is supposed to interpolate between the two values at its ends by the shorter of the two possible paths. A string passes through a plaquette if the net phase change around it is $\pm 2\pi$, rather than 0. The simplest implementation of this idea uses a lattice with tetrahedral cells, so that at most one string can pass into and out of each cell and there is no ambiguity about how they are connected.

With this algorithm, the lattice spacing determines the initial string density. It has usually been assumed that it should be identified with the correlation length ξ of the scalar Higgs or order-parameter field at the time of string formation. Then the initial string density (length per unit volume) is $1/\xi^2$ times a factor of say 1/4 representing the probability of finding a string in any given cell. However, an immediate question arises: what is the “time of formation”? Equivalently, at what temperature should we evaluate ξ ? The traditional answer to this question [4] has been to choose the Ginzburg temperature, T_G , which is defined so that above T_G , thermal fluctuations from the broken-symmetry state back over the central hump of the potential on the length scale ξ are frequent, while below it, they are rare. In transitions in weakly coupled theories, T_G is only slightly below the critical temperature T_c .

For several reasons, however, this cannot be the whole story. Firstly, it takes no account of the rate at which the transition proceeds. In thermal equilibrium, ξ diverges at the transition temperature, but if the transition proceeds at any finite rate any physically defined correlation length must remain finite, so it is far from obvious that the thermal equilibrium value of ξ is the right thing to use. To decide this question, we have to look at comparative rates. Secondly, the argument for using the Ginzburg temperature was that above T_G , because of the thermal fluctuations strings do not have any sort of permanent identity; one should not try to count them until that point has been reached. However, the thermal fluctuations mostly represent the transient appearance and disappearance of small loops of string, and are not necessarily relevant to the formation of long strings.

But perhaps the most cogent argument for re-thinking the criterion comes from exper-

iment. Zurek [5] suggested that a good test of the cosmic-string formation scenario might come from studies of the lambda transition in liquid helium. Hendry et al. [6] have recently performed such an experiment, using a pressure quench to take the system rapidly through the transition from normal to superfluid. The results certainly seem consistent with the picture of vortices evolving towards a scaling regime. (A similar picture has emerged from studies of the isotropic-to-nematic transition in liquid crystals [7].) In liquid helium, however, the Ginzburg temperature is quite far below the critical temperature. In fact, these experiments never reach the regime below T_G , so quite clearly it makes no sense to say that the initial scale of the vortices is determined by the correlation length at T_G .

In what follows we shall examine the various length scales in the problem and how they evolve with time. This will lead us to a new formulation of the criterion for estimating the string density.

II. THE GINZBURG LENGTH

It is useful to have a specific model in mind. We shall assume that the theory can be adequately described by a Landau-Ginzburg model, involving a scalar field ϕ . The free-energy density is assumed to be of the form

$$\mathcal{F} = |\dot{\phi}|^2 + |\nabla\phi|^2 - \frac{1}{2}m^2(T)|\phi|^2 + \frac{1}{8}\lambda|\phi|^4. \quad (1)$$

(We use relativistic normalization, but it would be easy to change to non-relativistic conventions.) The coefficient of $|\phi|^2$ vanishes at the critical temperature T_c and for $T < T_c$ but not too far below has the form

$$m^2(T) = m_0^2 \left(1 - \frac{T}{T_c}\right), \quad (2)$$

where $m_0^2 \sim \lambda T_c^2$. In the case of liquid helium, the Landau-Ginzburg model is known to be rather inaccurate, but it should still give a qualitatively reasonable description. We are not at the moment seeking anything better.

In thermal equilibrium at $T < T_c$, ϕ acquires a non-zero mean value, given by

$$|\phi|^2 = 2m^2(T)/\lambda \equiv \phi_{\text{eq}}^2(T). \quad (3)$$

The thermal-equilibrium correlation length ξ_c of ϕ is simply

$$\xi_c(T) = m^{-1}(T) \sim [\lambda T_c(T_c - T)]^{-1/2}. \quad (4)$$

This length scale defines the effective width of a string at temperature T .

The problem we want to address is the following. Given that the system goes through the phase transition at some known rate, how do we calculate the initial length scale ξ_s of the string network, and at what temperature T_s is this first possible? Here ξ_s is defined so that the string density at T_s is $1/\xi_s^2$. The idea is that we can then use the knowledge of ξ_s as initial data for numerical or analytic studies of string evolution.

It is also convenient to introduce another length scale, which we shall call the *Ginzburg length*, ξ_G . (The relation to the Ginzburg temperature will become clear shortly.) The

Ginzburg length is defined as the largest length scale on which thermal fluctuations from ϕ_{eq} back to $\phi = 0$ are probable. As we shall see, it is not necessarily correct to assume thermal equilibrium, but if we do the condition is simply

$$\xi_G^3 \delta \mathcal{F} = T, \quad (5)$$

where

$$\delta \mathcal{F} = \mathcal{F}(\phi = 0) - \mathcal{F}(\phi_{\text{eq}}) = \frac{m^4(T)}{2\lambda}. \quad (6)$$

Assuming that $\lambda \ll 1$, this yields

$$\xi_G \sim (\lambda T_c)^{-1/3} (T_c - T)^{-2/3}. \quad (7)$$

The Ginzburg length defines the “fuzziness” of the strings. Small loops of size up to ξ_G will continually appear and disappear. Long strings too will be subject to rapid fluctuations. Small fluctuating loops appearing near a long string may connect to it, causing it to wander in a random fashion. So it is difficult to say exactly where a long string is, within a distance ξ_G . Thermal fluctuations are liable to cause two long strings to intercommute if they come within a distance ξ_G of each other.

The *Ginzburg temperature*, T_G is simply the temperature at which $\xi_G = \xi_c$. From (4) and (7), we see that this requires

$$T_c - T_G \sim \lambda T_c, \quad (8)$$

a well-known result. Note that ξ_G decreases faster than ξ_c : for temperatures in the range $T_G < T < T_c$, $\xi_G > \xi_c$.

The assumption that thermal equilibrium is maintained is clearly not entirely correct. What we are interested in is the probability of a fluctuation from ϕ_{eq} to $\phi = 0$ on some given scale L . The thermal equilibrium properties of the system, such as ξ_c and ξ_G , are changing on a time scale of order $t - t_c$, where t_c is the time of the phase transition. So the thermal equilibrium calculation above is likely to be a good estimate provided that $\xi_G < t - t_c$. In the cosmological case, we have $T^2 t \sim M_P$, where M_P is the Planck mass. This inequality then yields

$$\frac{T_c - T}{T_c} > \frac{1}{\lambda^{1/5}} \left(\frac{T_c}{M_P} \right)^{3/5}. \quad (9)$$

We conclude that (7) should be reasonably reliable except *very* close to the transition.

For a condensed-matter system cooled through its transition temperature at a given rate dT/dt , the equivalent condition would be $\xi_G < v(t - t_c)$, where v is an appropriate characteristic speed, say the speed of second sound. This would give

$$\frac{T_c - T}{T_c} > \frac{1}{\lambda^{1/5}} \left(\frac{1}{v T_c^2} \left| \frac{dT}{dt} \right| \right)^{3/5}. \quad (10)$$

However, this is not directly applicable to the liquid-helium experiments, because temperature is not in fact the controlling variable. It would be more accurate to say that the transition proceeds because the sudden decrease of pressure leads to an increase of T_c .

III. THE STRING LENGTH SCALE

If the system is maintained in thermal equilibrium at a temperature below T_G , almost all strings will eventually disappear. The equilibrium abundance of loops of length L is determined by the Boltzmann factor $\exp[-\mu(T)L/T]$, where $\mu(T) \sim m^2(T)/\lambda$ is the energy per unit length of strings at temperature T . Thus the loop density at temperature T is exponentially suppressed for loop sizes greater than

$$L \sim \frac{\lambda T}{m^2(T)} \sim \frac{1}{T_c - T}. \quad (11)$$

For temperatures below the Ginzburg temperature, one finds that L is less than the string width.

At temperatures in the range $T_G < T < T_c$, there may be a significant equilibrium distribution of small loops, but no long strings would survive once thermal equilibrium had been reached. The crucial point, however, is that it takes a very long time (compared to the time required to establish thermal equilibrium in other respects) for long strings to disappear. Qualitatively, this is the reason for the appearance of a scaling solution. The time required for string structures on a given scale L to disappear increases with L , so that after some time only the structures with large L survive.

If we look at the system shortly after the phasetransition, when the temperature is still above T_G , we shall see violent fluctuations on small scales. As we noted above, it is difficult to say exactly where a long string is, to within a distance ξ_G . Nevertheless, viewed on a larger scale, it can be seen to have a more permanent existence. Long strings may indeed wander on a short time scale, but no small-scale fluctuation can make them disappear. Their large-scale configuration can change only if they encounter another long string and exchange partners with it. The long strings cannot be identified individually until their mean separation, ξ_s say, exceeds ξ_G .

How then can we estimate the string density, excluding the small transient loops? Ideally, we would like to be able to follow the process dynamically, starting from an initial state in the symmetric phase above the transition, but that is very hard to do. We adopt instead a more indirect approach, in which the relevant length scale is computed by a self-consistency argument.

We suppose that there is a temperature T_s , which we call the string-formation temperature, below which it is possible to identify strings, and try to estimate T_s by following the *subsequent* evolution of the strings. Of course, T_s must depend on the rate at which the transition proceeds, *i.e.*, on the expansion rate of the Universe.

Shortly after string formation, the strings are moving in a dense environment and are heavily damped. The force per unit length on a string moving with velocity v through this environment will be given by

$$f \sim n\sigma v_T \Delta p, \quad (12)$$

where n is the particle density, σ the linear cross section for string-particle scattering, v_T the thermal velocity and Δp the mean momentum transfer. In the familiar case of a thin string, we usually take $n \sim T^3$, $\sigma \sim T^{-1}$, $v_T \sim 1$ and $\Delta p \sim vT$, so that $f \sim T^3 v$. In our case, however, the strings are still very thick, in the sense that $\xi_c = m^{-1} \gg T^{-1}$. Thus

particles with thermal wavelengths are not scattered by the string, but simply pass through it unaffected. We should therefore include only particles with wavelengths comparable to or larger than the string width, *i.e.*, with momenta $k < \xi_c^{-1}$. Their density is $n \sim T/\xi_c^2$, and with $\sigma \sim \xi_c$, $\Delta p \sim v/\xi_c$, we get

$$f \sim v T m^2(T). \quad (13)$$

If the length scale of the long string configuration is ξ_s , then this damping force must be matched by the force due to the string tension, which on average is $\mu(T)/\xi_s$, where $\mu(T) \sim m^2(T)/\lambda$, so the typical string velocity will be

$$v \sim 1/(\lambda T_c \xi_s). \quad (14)$$

The characteristic time on which the length scale ξ_s will grow will be the time it takes for the string to move a distance ξ_s , namely ξ_s/v . Thus we expect

$$\frac{d\xi_s}{dt} \sim \frac{v}{\xi_s} \xi_s \sim \frac{1}{\lambda T_c \xi_s}. \quad (15)$$

So, unless ξ_s is initially very large, it will tend to grow as $(t - t_c)^{1/2}$. (This is also the growth law that has been deduced for the defects formed at phase transitions in various condensed-matter systems [8].) We can estimate the value of ξ_s as

$$\xi_s \sim \left(\frac{t - t_c}{\lambda T_c} \right)^{1/2}. \quad (16)$$

In the cosmological case, we can use the time-temperature relation to write this as

$$\xi_s \sim \left(\frac{M_P(T_c - T)}{\lambda T_c^4} \right)^{1/2}. \quad (17)$$

This calculation does *not* apply very close to T_c . In that region it is not possible to identify individual long strings, though one can use the methods of high-temperature field theory to follow the evolution of the string density, if strings are interpreted simply as loci of zeros of the Higgs field [9]. In the immediate vicinity of T_c the separation between zeros is typically $\xi_s \sim T_c^{-1}$. It then probably grows like some power of $t - t_c$ until we reach the range where the above calculation can be used.

We can now compare this length with the other length scales to see when the argument can be used. Clearly, the discussion only makes sense if the length scale of the long strings is larger than their width, *i.e.*, $\xi_s > \xi_c$. But in addition, we cannot really identify individual long strings until their length scale is larger than the Ginzburg length, so we also require that $\xi_s > \xi_G$. We can reasonably take the string-formation temperature to be the point at which both these inequalities are first satisfied. It is easy to check that

$$\xi_s = \xi_c \iff \frac{T_c - T}{T_c} \sim \left(\frac{T_c}{M_P} \right)^{1/2}. \quad (18)$$

whereas

$$\xi_s = \xi_G \iff \frac{T_c - T}{T_c} \sim \lambda^{1/7} \left(\frac{T_c}{M_P} \right)^{3/7}. \quad (19)$$

We are interested in the equality which occurs later, or at a lower value of T . Thus the relevant condition is (19) rather than (18) provided that

$$T_c/M_P < \lambda^2. \quad (20)$$

Except in cases when λ is very small, this condition will be satisfied for most of the relevant transitions. For GUT-scale transitions, it may be marginal, while for transitions at lower scales it will almost always hold. In any case, we are *never* likely to have a situation where $T_c/M_P \gg \lambda$.

If T_s is indeed given by (19), then the corresponding length scale at the time is

$$\xi_s \sim \xi_G \sim \frac{1}{\lambda^{3/7}} \left(\frac{M_P}{T_c} \right)^{2/7} \frac{1}{T_c}. \quad (21)$$

We can also now check the consistency of our calculation of ξ_G in the relevant temperature range. The required condition was (9). It is easy to check that this will be satisfied at the temperature T_s given by (19) if and only if the condition (20) is satisfied. Thus when ξ_G is the relevant scale length, our calculation of it was valid.

IV. CONCLUSIONS

There are several important length scales in this problem: the correlation length ξ_c which determines the width of strings; the Ginzburg length ξ_G which characterizes the extent of their “fuzziness” due to thermal fluctuations, and the typical separation ξ_s between the strings. Just after the phase transition, $\xi_G > \xi_s$; both are decreasing but ξ_G falls faster and eventually the two become equal at the Ginzburg temperature T_G .

The typical separation ξ_s evolves dynamically, increasing from an initial value $\sim T_c^{-1}$ at the transition. Individual strings can only be identified once ξ_s becomes larger than both ξ_c and ξ_G . By looking at the later evolution of ξ_s , we have estimated the temperature T_s at which this occurs. In almost all cases, the relevant criterion is (19). The corresponding length scale is (21). This essentially completes our task.

It would be interesting to apply a similar analysis to the case of the evolution following a pressure-induced quench in liquid helium. In that case, there could be interesting phenomena dependent on parameters such as the initial temperature from which the quench starts. We expect in particular that the statistical properties of the string network can be substantially modified if the quench starts very close to T_c .

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